

Averaging along foliated Lévy diffusions

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Abstract

This article studies the dynamics of the strong solution of a SDE driven by a discontinuous Lévy process taking values in a smooth foliated manifold with compact leaves. It is assumed that it is *foliated* in the sense that its trajectories stay on the leaf of their initial value for all times a.s.. Under a generic ergodicity assumption for each leaf, we determine the effective behaviour of the system subject to a small smooth perturbation of order $\varepsilon > 0$, which acts transversal to the leaves. The main result states that, on average, the transversal component of the perturbed SDE converges uniformly to the solution of a deterministic ODE as ε tends to zero. This transversal ODE is generated by the average of the perturbing vector field with respect to the invariant measures of the unperturbed system and varies with the transversal height of the leaves. We give upper bounds for the rates of convergence and illustrate these results for the random rotations on the circle. This article complements the results by Gargate and Ruffino for SDEs of Stratonovich type to general Lévy driven SDEs of Marcus type.

Keywords: averaging principle; Lévy diffusions on manifolds; foliated spaces; Marcus canonical equation

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1 Introduction

This article generalizes an averaging principle established for continuous semimartingales in González and Ruffino [5] to Lévy diffusions containing a jump component. The system under consideration is the strong solution of a stochastic differential equation (SDE) driven by a discontinuous Lévy noise with values in a smooth Riemannian manifold M equipped with a foliation structure \mathfrak{M} . That means there exists an equivalence relation on M , which defines a family of immersed submanifolds of constant dimension n , which cover M . The elements of \mathfrak{M} are called the leaves of the foliation. Moreover the solution flow of the SDE is assumed to respect the foliated structure \mathfrak{M} in the sense that each of the discontinuous solution paths of the SDE stays on the corresponding leaf of its initial condition for all times almost surely. We further assume the existence of a unique invariant measure for the SDE on each leaf.

If this system is perturbed by a smooth deterministic vector field εK transversal to the leaves with intensity $\varepsilon > 0$, the foliated structure of the solution is destroyed due to the appearance of

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a (smooth) transversal component in the trajectories. We study the effective behaviour of this transversal component in the limit as ε tends to 0.

The main idea is the following. Consider the solution along the rescaled time t/ε , its foliated component approximates the ergodic average behaviour for small ε . Hence the essential transversal behaviour is captured by an ODE for the transversal component driven by the vector field K instead of εK , which is averaged by the ergodic invariant measure on the leaves. Note that the intensity of the original perturbation εK cancels out by the time scaling t/ε . This is the result of Theorem 2.2 and will be referred to as an averaging principle. Our calculations here also determine upper bounds for the rates of convergence and a probabilistic robustness result.

The heuristics of an averaging principle consists in replacing the fine dynamical impact of a so-called fast variable on the dynamics of a so-called slow variable by its averaged statistical static influence. For references on the vast also classical literature on averaging for deterministic systems see e.g. the books by V. Arnold [2] and Saunders, Verhulst and Murdock [16] and the numerous citations therein. For stochastic systems among many others we mention the book by Kabanov and Pergamenschikov [6] and the references therein which gives an excellent overview on the subject. See also [3, 8]. An inspiration for this article also goes back to the work [12] by Li, where she established an averaging principle for the particular case of completely integrable (continuous) stochastic Hamiltonian systems. In Gonz  les and Ruffino [5] these results have been generalized to averaging principles for perturbations of Gaussian diffusions on foliated spaces. This article completes this result for general L  vy driven foliated diffusions.

The article is organized as follows. In the next Section we present the main result and an example to illustrate the averaging phenomenon. Section 3 is dedicated to the fundamental technical Proposition 3.1, where the stochastic Marcus integral technique is applied and whose estimates turn out to be the basis for the rates of convergence of the main theorem. Section 4 deals with the averaging on the leaves. In Section 5 we prove the main theorem. Further details of the calculations in the example of Section 2.3 are provided in an Appendix.

2 Object of study and main results

2.1 The set up

Let M be a connected smooth Riemannian manifold with an n -dimensional smooth foliation \mathfrak{M} . Given an initial condition $x_0 \in M$, we assume that there exists a bounded neighbourhood $U \subset M$ of the corresponding compact leaf L_{x_0} such that there exists a diffeomorphism $\varphi : U \rightarrow L_{x_0} \times V$, where $V \subset \mathbb{R}^d$ is a connected open set containing the origin. The neighbourhood U is taken small enough such that the derivatives of φ are bounded (say, U is precompact in M). The second coordinate, called the vertical coordinates of a point $q \in U$ will be denoted by the projection $\Pi : U \rightarrow V$ with $\Pi(q) \in V$, i.e. $\varphi(q) = (u, \Pi(q))$ for some $u \in L_{x_0}$. Hence for any fixed $v \in V$, the inverse image $\Pi^{-1}(v)$ is the compact leaf L_x , where x is any point in U such that the vertical projection satisfies $\Pi(x) = v$. The components of the vertical projection are denoted by

$$\Pi(q) = \left(\Pi_1(q), \dots, \Pi_d(q) \right) \in V \subset \mathbb{R}^d,$$

for any $q \in U$.

We are interested in a Lévy driven SDE with values in M embedded in an Euclidean space, whose solutions respect the foliation. Since such a solution necessarily satisfies a canonical Marcus equation, also known as generalized Stratonovich equation in the sense of Kurtz, Pardoux and Protter [10], we consider the stochastic differential equation

$$dX_t = F_0(X_t)dt + F(X_t) \diamond dZ_t, \quad X_0 = x_0, \quad (1)$$

which consists of the following components.

1. Let $F \in \mathcal{C}^1(M; L(\mathbb{R}^r; T\mathfrak{M}))$, such that the map $x \mapsto F(x)$ is \mathcal{C}^1 such that for each $x \in M$ the linear map $F(x)$ sends a vector $z \in \mathbb{R}^r \mapsto F(x)z \in T_x L_x$ to the respective tangent space of the leaf. Furthermore we assume that F and $(DF)F$ are globally Lipschitz continuous on M with common Lipschitz constant $\ell > 0$. We write $F_i(x) = F(x)e_i$, for e_i the canonical basis of \mathbb{R}^r .
2. Here $Z = (Z_t)_{t \geq 0}$ with $Z_t = (Z_t^1, \dots, Z_t^r)$ is a Lévy process in \mathbb{R}^r with respect to a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with characteristic triplet $(0, \nu, 0)$. It is assumed that the filtration satisfies the usual conditions in the sense of Protter [13]. It is a consequence of the Lévy-Itô decomposition of Z that Z is a pure jump process with respect to a Lévy measure $\nu : \mathcal{B}(\mathbb{R}^r) \rightarrow [0, \infty]$ which satisfies $\nu\{0\} = 0$ and permits the existence of an exponential moment of order $\kappa > \ell$ given by

$$\int_{\mathbb{R}^r} (e^{\kappa\|y\|} \wedge \|y\|^2) \nu(dy) < \infty. \quad (2)$$

where ℓ the Lipschitz constant of the function $x \mapsto F(x)$ introduced in point 1. For details we refer to the monographs of Sato [14] or Applebaum [1].

3. Equation (1) is defined as

$$X_t = x_0 + \int_0^t F_0(X_s)ds + \int_0^t F(X_{s-})dZ_s + \sum_{0 < s \leq t} (\Phi^{F\Delta_s Z}(X_{s-}) - X_{s-} - F(X_{s-})\Delta_s Z), \quad (3)$$

where the function $\Phi^{Fz}(x) = Y(1, x; Fz)$ and $Y(t, x; Fz)$ stands for the solution of the ordinary differential equation

$$\frac{d}{d\sigma} Y(\sigma) = F(Y(\sigma))z, \quad \text{with initial condition } Y(0) = x \in M, \quad z \in \mathbb{R}^r. \quad (4)$$

This article studies the situation where a foliated SDE is perturbed by a transversal smooth vector field εK with $\varepsilon > 0$ in the limit for $\varepsilon \searrow 0$.

4. For $K : M \rightarrow TM$ a smooth, globally Lipschitz continuous vector field we denote by X^ε , $\varepsilon > 0$ the solution of the perturbed system

$$dX_t^\varepsilon = F_0(X_t^\varepsilon)dt + F(X_t^\varepsilon) \diamond dZ_t + \varepsilon K(X_t^\varepsilon)dt, \quad X_0^\varepsilon = x_0. \quad (5)$$

Theorem 2.1 ([10], Theorem 3.2 and 5.1) *1. Under the previous assumptions notably item 1.- 3., there is a unique semimartingale X which is a strong global solution of (1) in the sense of equation (3). It has a càdlàg version and is a (strong) Markov process.*

2. Under the previous assumptions in particular item 1.-4., there is a unique semimartingale X^ε which is a strong global solution of equation (5) in the sense of equation (3), where F_0 is replaced by $F_0 + \varepsilon K$. The perturbed solution X^ε has càdlàg paths almost surely and is a (strong) Markov process.

With the previously mentioned embedding results in mind we are now in the position to apply Proposition 4.3 in Kurtz, Pardoux and Protter [10], which states the following support theorem. Under the aforementioned conditions $\mathbb{P}(X_0 \in M) = 1$, it implies that $\mathbb{P}(X_t \in M, \forall t \geq 0) = 1$. The same result applied again on the leaves yields in particular that each solution is *foliated* in the sense that it stays on the leaf of its initial condition, i.e. $\mathbb{P}(X_0 \in L_{x_0}) = 1$ implies that $\mathbb{P}(X_t \in L_{x_0}, \forall t \geq 0) = 1$. We shall call an SDE of the type (1) which admits a foliated solution a *foliated stochastic differential equation*.

2.2 The main results

We assume that each leaf $L_q \in \mathfrak{M}$ passing through $q \in M$ has associated a unique invariant measure μ_q of the unperturbed foliated system (1) with initial condition $x_0 = q$.

Let $\Psi : M \rightarrow \mathbb{R}$ be a differentiable function. We define the average of Ψ with respect to μ_q in the following way. If v is the vertical coordinate of q , that is $\varphi(q) = (u, v)$, we define

$$Q^\Psi(v) := \int_{L_q} \Psi(y) \mu_q(dy). \quad (6)$$

With respect to the coordinates given by φ , the perturbing vector field K is written as

$$d\Pi(K) = (d\Pi_1(K), \dots, d\Pi_d(K)).$$

Hypothesis 1: For $i = 1, \dots, d$ the function

$$v \mapsto Q^{d\Pi_i K}(v) \quad (7)$$

is globally Lipschitz continuous.

This ensures in particular that for each $w \in \mathbb{R}^d$ the ordinary differential equation

$$\frac{dw}{dt}(t) = (Q^{d\Pi_1(K)}, \dots, Q^{d\Pi_d(K)})(w(t)), \quad w(0) = w \quad (8)$$

has a unique global solution.

We are going to consider the average on the leaves of each real valued functions $d\Pi_i(K)$, with $i = 1, \dots, d$. In general, there is no standard rate of convergence in the ergodic theorem, see e.g. Krengel [9], Kakutani and Petersen [7]. We are going to consider a prescribed rate of converge in time average, following the classical approach as in Freidlin and Wentzell [4, Chap. 7.9], where they deal with an averaging principle with convergence in probability.

Hypothesis 2: For any $x_0 \in M$ and $p \geq 2$ there exists a positive, bounded, decreasing function $\eta : [0, \infty) \rightarrow [0, \infty)$ which estimates the rate of convergence of the ergodic unperturbed dynamic in each leaf L_{x_0} in the sense that for all $i \in \{1, \dots, d\}$

$$\left(\mathbb{E} \left| \frac{1}{t} \int_0^t d\Pi_i K(X_s(x_0)) ds - Q^{d\Pi_i(K)}(\Pi(x_0)) \right|^p \right)^{\frac{1}{p}} \leq \eta(t), \quad \text{for all } t \geq 0. \quad (9)$$

For $\varepsilon > 0$ and $x_0 \in M$ let τ^ε be the first exit time of the solution $X^\varepsilon(x_0)$ of equation (5) from the aforementioned foliated coordinate neighborhood U .

Theorem 2.2 *Assume that the unperturbed foliated system (1) on M satisfies Hypothesis 1 and 2. Let w be the solution of the deterministic ODE in the transversal component $V \subset \mathbb{R}^n$.*

$$\frac{dw}{dt}(t) = (Q^{d\Pi_1(K)}, \dots, Q^{d\Pi_d(K)})(w(t)), \quad (10)$$

with initial condition $w(0) = \Pi(x_0) = 0$. Let T_0 be the time that $w(t)$ reaches the boundary of V .

Then we have that:

1. For all $0 < t < T_0$, $\beta \in (0, \frac{1}{2})$, $p \geq 2$ and $\lambda < 1$

$$\left(\mathbb{E} \left[\sup_{s \leq t} |\Pi(X_{\frac{s}{\varepsilon} \wedge \tau^\varepsilon}^\varepsilon) - w(s)|^p \right] \right)^{\frac{1}{p}} \leq t \left[\varepsilon^\lambda h(t, \varepsilon) + \eta \left(t |\ln \varepsilon|^{\frac{2\beta}{p}} \right) \exp \{Ct\} \right],$$

where $h(t, \varepsilon)$ is continuous and converges to zero, when ε or t do so and C is a positive constant.

2. For $\gamma > 0$, let

$$T_\gamma := \inf \{t > 0 \mid \text{dist}(w(t), \partial V) \leq \gamma\}.$$

The exit times of the two systems satisfy the estimates

$$\mathbb{P}(\varepsilon \tau^\varepsilon < T_\gamma) \leq \gamma^{-p} T_\gamma^p \left[\varepsilon^\lambda h(T_\gamma, \varepsilon) + \eta \left(T_\gamma |\ln \varepsilon|^{-\frac{2\beta}{p}} \right) e^{CT_\gamma} \right]^p.$$

The second part of the theorem above guarantees the robustness of the averaging phenomenon in transversal direction.

2.3 Example. Perturbed random rotations: The Gamma process on the circle

As a simple but illustrative example of the phenomenon, consider $M = \mathbb{R}^3 \setminus \{(0, 0, z), z \in \mathbb{R}\}$ with the 1-dimension horizontal circular foliation of M where the leaf passing through a point $q = (x, y, z)$ is given by the (non-degenerate) horizontal circle at height z :

$$L_q = \{(\sqrt{x^2 + y^2} \cos \theta, \sqrt{x^2 + y^2} \sin \theta, z), \quad \theta \in [0, 2\pi)\}.$$

This foliation (cf. [5]) is an example where the transversal space is richer than the leaves themselves, hence the range of the impact for different perturbations is relatively large.

Let the process $Z = (Z_t)_{t \geq 0}$ be a Gamma process in \mathbb{R} with characteristic triplet $(0, \nu, 0)$, where $\nu(dy) = \frac{e^{-\theta|y|}}{|y|}$ is the corresponding Lévy jump measure with a rate parameter $\theta > 0$. It satisfies the integrability condition $\int_{\mathbb{R}} (e^{\kappa|y|} \wedge |z|^2) \nu(dz) < \infty$ required in (2). The Lévy-Itô decomposition of Z yields the almost sure decomposition

$$Z_t = \int_0^t \int_{|y| \leq 1} y \tilde{N}(ds, dy) + \int_{|y| > 1} y N(ds, dy), \quad (11)$$

where N is the random Poisson measures with intensity measure $dt \otimes \nu$. \tilde{N} denotes its compensated counterpart. Consider the foliated linear SDE on M consisting of random rotations:

$$dX_t = \Lambda X_t \diamond dZ_t, \quad X_0 = q_0 = (x_0, y_0, z_0), \quad (12)$$

where

$$\Lambda = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Equation (12) is defined as follows: First note that $\Lambda^2 = \text{diag}(-1, -1, 0)$. Secondly, note that for $z \in \mathbb{R}$, $z \neq 0$ the solution flow Φ of the equation

$$\frac{d}{d\sigma} Y(\sigma) = F(Y(\sigma))z, \quad Y(0) = q \quad \text{where } F(\bar{q}) = \Lambda \bar{q}$$

is obtained by a simple calculation as

$$\Phi^{Fz}(q) = Y(1; q) = \begin{pmatrix} x \cos(z) - y \sin(z) \\ x \sin(z) + y \cos(z) \\ z \end{pmatrix},$$

such that

$$X_t = q_0 + \int_0^t \Lambda X_{s-} z \tilde{N}(ds, dz) + \sum_{0 < s \leq t} (\Phi^{F\Delta_s Z}(X_{s-}) - X_{s-} - F(X_{s-})\Delta_s Z).$$

We ignore for the moment the constant third component $X_t^3 = z_0$ for all $t \geq 0$ almost surely and write $\bar{X} = (X^1, X^2)$ for convenience. Using the chain rule of the Marcus integral, as stated in Proposition 4.2 in [10], we verify for $\chi(x, y) := x^2 + y^2$ we get

$$d\chi(\bar{X}_t) = -2\bar{X}_{t-} \Lambda \bar{X}_{t-} \diamond dZ_t = 0. \quad (13)$$

In fact, \bar{X}^ε can be defined equally as the projection of Z on the unit circle. If we identify the plane, where \bar{X} takes its values, with the complex plane \mathbb{C} , one verifies easily that

$$\bar{X}_t = e^{iZ_t}.$$

In fact we obtain the well-known Lévy-Chinchine representation of the characteristic function for any $p \in \mathbb{R}$:

$$\begin{aligned} \mathbb{E}[\bar{X}_t^p] &= \mathbb{E}[e^{ipZ_t}] = \exp(t\Psi(p)), \\ \Psi(p) &= \int_{\mathbb{R}^d} (e^{ipy} - 1 - iyp\mathbf{1}\{|y| \leq 1\}) d\nu(y). \end{aligned}$$

It is a direct consequence of Lemma 6.1 that for any uniformly distributed random variable on a circle in the complex plane centered in the origin we have $Z_t \xrightarrow{d} U$ as $t \rightarrow \infty$. The invariant measures μ_q in the leaves L_q passing through points $q \in M$ are therefore given by normalized Lebesgue measures in the circle L_q centered in 0 with radius $|(x, y)|$. We investigate the effective behaviour of a small transversal perturbation of order ε :

$$dX_t^\varepsilon = \Lambda X_t^\varepsilon \diamond dZ_t + \varepsilon K(X_t^\varepsilon) dt$$

with initial condition $q_0 = (1, 0, 0)$. In this example we shall consider two classes of perturbing vector field K .

(A) Constant perturbation εK . Assume that the perturbation is given by a constant vector field $K = (K_1, K_2, K_3) \in \mathbb{R}^3$ with respect to Euclidean coordinates in M . Then, the horizontal average of the radial component

$$Q^{d\Pi_r K}(r_0, z_0) = \int_{L_q} \langle (K_1, K_2)^T, y \rangle d\mu_q(y) = 0, \quad q = (\theta_0, r_0, z_0)$$

and the vertical z -component is constant $Q^{d\Pi_z K} = K_3$. We verify (9) and obtain with the help of (13) the rate function $\eta \equiv 0$. In fact trivially:

$$\left(\mathbb{E} \left[\left| \frac{1}{t} \int_0^t (d\Pi_r K)(X_s) ds - Q^{d\Pi_r K}(r_0, z_0) \right|^p \right] \right)^{\frac{1}{p}} = 0,$$

and

$$\left(\mathbb{E} \left[\left| \frac{1}{t} \int_0^t (d\Pi_z K)(X_s) ds - Q^{d\Pi_z K}(r_0, z_0) \right|^p \right] \right)^{\frac{1}{p}} = K_3 - K_3 = 0.$$

Hence the transversal component in Theorem 2.2 for initial condition $q_0 = (1, 0, 0)$ is given by $w(t) = (1, K_3 t)$ for all $t \geq 0$. Theorem 2.2 establishes a minimum rate of convergence to zero of the difference between each of the transversal components. Hence for the radial component of the perturbed systems $w_1(t) \equiv 1$ and $\Pi_r(X_{\frac{t}{\varepsilon} \wedge \tau^\varepsilon})$ it holds that, for $p \geq 2$ and $\gamma \in (0, 1)$

$$\left[\mathbb{E} \left(\sup_{s \leq t} \left| \Pi_r(X_{\frac{t}{\varepsilon} \wedge \tau^\varepsilon}) - 1 \right|^p \right) \right]^{\frac{1}{p}} \leq \varepsilon^\gamma t.$$

For the second transversal component, we have that

$$|\Pi_z(X_{\frac{t}{\varepsilon} \wedge \tau^\varepsilon}) - w_2(t)| \equiv 0$$

for all $t \geq 0$ and the convergence of the theorem is trivially verified.

(B) Linear perturbation $\varepsilon K(x, y, z) = \varepsilon(x, 0, 0)$. For the sake of simplicity, we consider a one dimensional and horizontal linear perturbation, which in this case can be written in the form $K(x, y, z) = (x, 0, 0)$. The z -coordinate average vanishes trivially. For the radial component, we have that $d\Pi_r K(q) = r \cos^2(\theta)$, where θ is the angular coordinate of $q = (\theta, r, z)$ whose distance to the z -axis (radial coordinate) is r . Hence the average with respect to the invariant measure on the leaves is given by

$$Q^{d\Pi_r K}(\theta, r, z) = \frac{1}{2\pi} \int_0^{2\pi} r \cos^2(\theta) d\theta = \frac{r}{2}$$

for leaves L_q with radius r . We verify the convergence (9) of Hypothesis 2 for the radial component and $p = 2$. Elementary calculations, which can be found in the Appendix after Lemma 6.1, show that

$$\mathbb{E} \left[\left| \frac{1}{t} \int_0^t d\Pi_r K(Z_s, 0, 0) ds - Q^{d\Pi_r K}(q_0) \right|^2 \right]$$

$$\begin{aligned}
&= \mathbb{E} \left[\left| \frac{1}{t} \int_0^t r \cos^2(Z_s) ds - \frac{r}{2} \right|^2 \right] \\
&= \left(\frac{r}{t} \right)^2 2 \int_0^t \int_0^\sigma \mathbb{E} \left[\cos^2(Z_s) \cos^2(Z_\sigma) \right] ds d\sigma + \frac{r^2}{t} \int_0^t \frac{1}{2} \exp(-Cs) ds - \frac{r^2}{4},
\end{aligned}$$

where the first term can be estimated by constants $a, b > 0$

$$\left(\frac{r}{t} \right)^2 2 \int_0^t \int_0^\sigma \mathbb{E} \left[\cos^2(Z_s) \cos^2(Z_\sigma) \right] ds d\sigma \leq \left(\frac{r}{t} \right)^2 \frac{1}{4} (a + bt + t^2) \xrightarrow{t \nearrow \infty} \frac{r^2}{4}.$$

Combining the previous two results and taking the square root, the rate of convergence is of order $\eta(t) = C/\sqrt{t}$ for a positive constant C as $t \nearrow \infty$.

For an initial value $q_0 = (x_0, y_0, z_0) = (r_0 \cos(u_0), r_0 \sin(u_0), z_0)$ the transversal system stated in Theorem 2.2 is then $w(t) = (e^{\frac{t}{2}} r_0, z_0)$. Hence the result guarantees that the radial part $\Pi_r(X_{\frac{t}{\varepsilon} \wedge \tau^\varepsilon}^\varepsilon)$ must have a behaviour close to the exponential $e^{\frac{t}{2}}$ in the sense that

$$\left[\mathbb{E} \left(\sup_{s \leq t} \left| \Pi_r(X_{\frac{s}{\varepsilon} \wedge \tau^\varepsilon}^\varepsilon) - e^{\frac{s}{2}} \right|^2 \right) \right]^{\frac{1}{2}} \leq Ct\varepsilon^\lambda + C\sqrt{t} \exp(Ct) |\ln \varepsilon|^{-\frac{\beta}{p}},$$

tends to zero when $\varepsilon \searrow 0$.

3 Transversal perturbations

Next proposition gives information on the order of which the perturbed trajectories approach the unperturbed trajectories when ε goes to zero.

Proposition 3.1 *For any Lipschitz test function $\Psi : M \rightarrow \mathbb{R}$ there exist positive constants k_1, k_2 such that for all $T \geq 0$ we have*

$$\left(\mathbb{E} \left[\sup_{t \leq T \wedge \tau^\varepsilon} |\Psi(X_t^\varepsilon(x_0)) - \Psi(X_t(x_0))|^p \right] \right)^{\frac{1}{p}} \leq k_1 \varepsilon T \exp(k_2 T^p). \quad (14)$$

The constants k_1 and k_2 depend on the upper bounds of the norms for the perturbing vector field K in U , on the Lipschitz coefficient of Ψ and on the derivatives of the vector fields F_0, F_1, \dots, F_r with respect to the coordinate system.

Proof: First we rewrite X^ε and X , the solutions of equation (1) and (5), in terms of the coordinates given by the diffeomorphism φ :

$$(u_t, v_t) := \varphi(X_t) \quad \text{and} \quad (u_t^\varepsilon, v_t^\varepsilon) := \varphi(X_t^\varepsilon).$$

Exploiting the regularities of Ψ and φ we obtain

$$\begin{aligned}
|\Psi(X_t^\varepsilon) - \Psi(X_t)| &= |\Psi \circ \varphi^{-1}(u_t^\varepsilon, v_t^\varepsilon) - \Psi \circ \varphi^{-1}(u_t, v_t)| \\
&\leq C_0 |u_t^\varepsilon - u_t, v_t^\varepsilon - v_t| \leq C_0 (|u_t^\varepsilon - u_t| + |v_t^\varepsilon - v_t|).
\end{aligned} \quad (15)$$

for $C_0 := \text{Lip}(\psi) \sup_{y \in U} |\varphi^{-1}(y)|$. The proof of the statement consists in calculating estimates for each summand on the right hand side of equation above. We define

$$\mathfrak{F}_i := (D\varphi) \circ F_i \circ \varphi^{-1} \quad \text{for } i = 0, \dots, n,$$

$$\mathfrak{K} := (D\varphi) \circ K \circ \varphi^{-1},$$

which all together with their derivatives are bounded. Considering the components in the image of φ we have:

$$\mathfrak{K} = (\mathfrak{K}_H, \mathfrak{K}_V),$$

with $\mathfrak{K}_H \in TL_{x_0}$ and $\mathfrak{K}_V \in TV \simeq \mathbb{R}^d$. The chain rule proved in Theorem 4.2 of [10] yields for equation (5) the following form in φ coordinates, written componentwise as

$$du_t^\varepsilon = \mathfrak{F}_0(u_t^\varepsilon, v_t^\varepsilon)dt + \mathfrak{F}(u_t^\varepsilon, v_t^\varepsilon) \diamond dZ_t + \varepsilon \mathfrak{K}_H(u_t^\varepsilon, v_t^\varepsilon)dt \quad \text{with } u_t^\varepsilon \in L_{x_0}, \quad (16)$$

$$dv_t^\varepsilon = \varepsilon \mathfrak{K}_V(u_t^\varepsilon, v_t^\varepsilon)dt \quad \text{with } v_t^\varepsilon \in V. \quad (17)$$

We start with estimates on the transversal components $|v^\varepsilon - v|$. Let $|K|$ and hence $|\mathfrak{K}|$ be bounded by a universal constant, $C_1 > 0$, say. Therefore equation (17) yields the estimate

$$\sup_{t \leq T \wedge \tau^\varepsilon} |v_t^\varepsilon - v_t| \leq \varepsilon \sup_{t \leq T \wedge \tau^\varepsilon} \int_0^t |\mathfrak{K}_V(u_s^\varepsilon, v_s^\varepsilon)| ds \leq C_1 \varepsilon T. \quad (18)$$

We continue with estimates on the difference of the ‘horizontal’ component. Recall that for $t < \tau^\varepsilon$ formally

$$u_t^\varepsilon - u_t = \int_0^t (\mathfrak{F}_0(u_s^\varepsilon, v_s^\varepsilon) - \mathfrak{F}_0(u_s, v_s))ds + \int_0^t (\mathfrak{F}(u_s^\varepsilon, v_s^\varepsilon) - \mathfrak{F}(u_s, v_s)) \diamond dZ_s + \varepsilon \int_0^t \mathfrak{K}_H(u_s^\varepsilon, v_s^\varepsilon)ds.$$

This equality is defined as

$$\begin{aligned} u_t^\varepsilon - u_t &= \int_0^t [\mathfrak{F}_0(u_s^\varepsilon, v_s^\varepsilon) - \mathfrak{F}_0(u_s, v_s)]ds \\ &+ \int_0^t [\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})]dZ_s \\ &+ \sum_{0 < s \leq t} [(\Phi^{\mathfrak{F}\Delta_s Z}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \Phi^{\mathfrak{F}\Delta_s Z}(u_{s-}, v_{s-})) \\ &\quad - (u_{s-}^\varepsilon - u_{s-}) - (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-}))\Delta_s Z] \\ &+ \varepsilon \int_0^t \mathfrak{K}_H(u_s^\varepsilon, v_s^\varepsilon)ds. \end{aligned}$$

Since $p \geq 2$ this leads to

$$\begin{aligned} |u_t^\varepsilon - u_t|^p &\leq 4^{p-1} \left| \int_0^t \mathfrak{F}_0(u_s^\varepsilon, v_s^\varepsilon) - \mathfrak{F}_0(u_s, v_s)ds \right|^p + 4^{p-1} C_1^p \varepsilon^p t^p \\ &+ 4^{p-1} \left| \int_0^t [\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})]dZ_s \right|^p \\ &+ 4^{p-1} \left| \sum_{0 < s \leq t} \Phi^{\mathfrak{F}\Delta_s Z}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \Phi^{\mathfrak{F}\Delta_s Z}(u_{s-}, v_{s-}) - (u_{s-}^\varepsilon - u_{s-}^i) \right. \\ &\quad \left. - (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-}))\Delta_s Z \right|^p. \end{aligned} \quad (19)$$

We now estimate the terms of the right-hand side in (19). The first term on the right-hand side is dominated by Jensen’s inequality and equation (18)

$$\left| \int_0^t \mathfrak{F}_0(u_s^\varepsilon, v_s^\varepsilon) - \mathfrak{F}_0(u_s, v_s)ds \right|^p \leq \left(\int_0^t C_2 |u_s^\varepsilon - u_s, v_s^\varepsilon - v_s| ds \right)^p$$

$$\begin{aligned}
&\leq C_2^p \left(\int_0^t (|u_s^\varepsilon - u_s| + |v_s^\varepsilon - v_s|) ds \right)^p \\
&\leq C_2^p (2t)^{p-1} \left(\int_0^t |u_s^\varepsilon - u_s|^p ds + \int_0^t |v_s^\varepsilon - v_s|^p ds \right) \\
&\leq C_2^p (2t)^{p-1} \left(\int_0^t |u_s^\varepsilon - u_s|^p ds + C_1^p \varepsilon^p t^{p+1} \right) \\
&\leq C_2^p (2t)^{p-1} \int_0^t |u_s^\varepsilon - u_s|^p ds + (2C_1 C_2)^p t^{2p} \varepsilon^p.
\end{aligned}$$

The term in the second line has the following representation with respect to the random Poisson measure associated to Z

$$\int_0^t [\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})] dZ_s = \int_0^t \int_{\mathbb{R}^r \setminus \{0\}} [\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})] z \tilde{N}(ds, dz).$$

By Kunita's first inequality for the supremum of integrals with respect to the compensated random Poisson measure integrals, as stated for instance in Theorem 4.4.23 in [1], and inequality (18) we obtain

$$\begin{aligned}
&\mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \int_{\mathbb{R}^r} [\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})] z \tilde{N}(ds, dz) \right|^p \right] \\
&\leq C_3 \left(\int_{\mathbb{R}^r} \|z\|^2 \nu(dz) \right)^{p/2} \mathbb{E} \left[\left(\int_0^T |\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})|^2 ds \right)^{p/2} \right] \\
&\quad + C_3 \int_{\mathbb{R}^r} \|z\|^p \nu(dz) \mathbb{E} \left[\left(\int_0^T |\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})|^p ds \right) \right] \\
&\leq (2C_2)^p C_3 (C_4 T^{\frac{p}{2}-1} + C_5) \int_0^T \mathbb{E} \left[\sup_{s \in [0, t]} |u_s^\varepsilon - u_s|^p \right] ds + (2C_1 C_2)^p C_3 (C_4 + C_5) T^{p+1} \varepsilon^p \quad (20) \\
&\leq (2C_2)^p C_3 (C_4 + C_5) (T^{\frac{p}{2}-1} + 1) \int_0^T \mathbb{E} \left[\sup_{s \in [0, t]} |u_s^\varepsilon - u_s|^p \right] ds + (2C_1 C_2)^p C_3 (C_4 + C_5) (T^{\frac{3p}{2}} + T^{p+1}) \varepsilon^p. \quad (21)
\end{aligned}$$

Note that C_4 and C_5 are finite due to the existence of an exponential moment (2). Since the vector fields \mathfrak{F} and $(D\mathfrak{F})\mathfrak{F}$ are globally Lipschitz continuous, we apply the estimates in Lemma 3.1 of [10], which yields a constant $C_6 = C_6(p) > 0$, then apply once again (18)

$$\begin{aligned}
\mathfrak{J}_t &:= \left| \sum_{0 < s \leq t} \Phi^{\mathfrak{F} \Delta_s Z}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \Phi^{\mathfrak{F} \Delta_s Z}(u_{s-}, v_{s-}) \right. \\
&\quad \left. - (u_{s-}^\varepsilon - u_{s-}, v_{s-}^\varepsilon - v_{s-}) - (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})) \Delta_s Z \right|^p \\
&\leq \left(C_6 \sum_{0 < s \leq t} |(u_{s-}^\varepsilon - u_{s-}, v_{s-}^\varepsilon - v_{s-})| e^{C_6 \|\Delta_s Z\|} \|\Delta_s Z\|^2 \right)^p \\
&\leq (C_6)^p \left(\sum_{0 < s \leq t} (|u_{s-}^\varepsilon - u_{s-}| + |v_{s-}^\varepsilon - v_{s-}|) e^{C_6 \|\Delta_s Z\|} \|\Delta_s Z\|^2 \right)^p \\
&\leq (2C_6)^p \left[\left(\sum_{0 < s \leq t} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|\Delta_s Z\|} \|\Delta_s Z\|^2 \right)^p + \left(C_1 \varepsilon t \sum_{0 < s \leq t} e^{C_6 \|\Delta_s Z\|} \|\Delta_s Z\|^2 \right)^p \right]. \quad (22)
\end{aligned}$$

We go over to the representation with the random Poisson measure. For the first summand we obtain

$$\begin{aligned}
& \sum_{0 < s \leq t} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|\Delta_s Z\|} \|\Delta_s Z\|^2 \\
&= \int_0^t \int_{\|y\| \leq 1} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|y\|} \|y\|^2 \tilde{N}(ds dy) + \int_0^t \int_{\|y\| \leq 1} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|y\|} \|y\|^2 \nu(dy) ds \\
&\quad + \int_0^t \int_{\|y\| > 1} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|y\|} \|y\|^2 N(ds dy) \\
&= \int_0^t \int_{\mathbb{R}^r} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|y\|} \|y\|^2 \tilde{N}(ds dy) + \int_0^t \int_{\mathbb{R}^r} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|y\|} \|y\|^2 \nu(dy) ds \tag{23}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{0 < s \leq t} e^{C_6 \|\Delta_s Z\|} \|\Delta_s Z\|^2 &= \int_0^t \int_{\|y\| \leq 1} e^{C_6 \|y\|} \|y\|^2 \tilde{N}(ds dy) + \int_0^t \int_{\|y\| \leq 1} e^{C_6 \|y\|} \|y\|^2 \nu(dy) ds \\
&\quad + \int_0^t \int_{\|y\| > 1} e^{C_6 \|y\|} \|y\|^2 N(ds dy) \\
&= \int_0^t \int_{\mathbb{R}^r} e^{C_6 \|y\|} \|y\|^2 \tilde{N}(ds dy) + \int_0^t \int_{\mathbb{R}^r} e^{C_6 \|y\|} \|y\|^2 \nu(dy) ds. \tag{24}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}[\mathfrak{J}_t] &\leq (2C_6)^p \left\{ \mathbb{E} \left[\left| \int_0^t \int_{\mathbb{R}^r} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|y\|} \|y\|^2 \tilde{N}(ds dy) \right|^p \right] \right. \\
&\quad \left. + \mathbb{E} \left[\left| \int_0^t \int_{\mathbb{R}^r} |u_{s-}^\varepsilon - u_{s-}| e^{C_6 \|y\|} \|y\|^2 \nu(dy) ds \right|^p \right] \right\} \\
&\quad + (2C_1 C_6 \varepsilon t)^p \left\{ \mathbb{E} \left[\left| \int_0^t \int_{\mathbb{R}^r} e^{C_6 \|y\|} \|y\|^2 \tilde{N}(ds dy) \right|^p \right] + \left| \int_0^t \int_{\mathbb{R}^r} e^{C_6 \|y\|} \|y\|^2 \nu(dy) ds \right|^p \right\} \\
&=: J_1 + J_2 + J_3 + J_4. \tag{25}
\end{aligned}$$

Kunita's first theorem [1] provides a constant $C_7 = C_7(p)$ for the estimate of the p -th moment for integrals with respect to compensated random Poisson measure in J_1 and J_4 .

$$\begin{aligned}
J_1 &\leq (2C_6 C_7)^p \left\{ \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^r} |u_{s-}^\varepsilon - u_{s-}|^p e^{C_6 p \|y\|} \|y\|^{2p} \nu(dy) ds \right] \right. \\
&\quad \left. + \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^r} |u_{s-}^\varepsilon - u_{s-}|^2 e^{C_6 2 \|y\|} \|y\|^4 \nu(dy) ds \right)^{\frac{p}{2}} \right] \right\} \\
&\leq (2C_6 C_7)^p \left\{ \int_{\mathbb{R}^r} e^{C_6 p \|y\|} \|y\|^{2p} \nu(dy) \mathbb{E} \left[\int_0^t |u_{s-}^\varepsilon - u_{s-}|^p ds \right] \right. \\
&\quad \left. + \left(\int_{\mathbb{R}^r} e^{2C_6 \|y\|} \|y\|^4 \nu(dy) \right)^{\frac{p}{2}} \mathbb{E} \left[\left(\int_0^t |u_{s-}^\varepsilon - u_{s-}|^2 ds \right)^{\frac{p}{2}} \right] \right\} \\
&\leq (2C_6 C_7)^p (C_8 + C_9) (T^{\frac{p}{2}-1} + 1) \int_0^T \mathbb{E} \left[\sup_{s \in [0, t]} |u_{s-}^\varepsilon - u_{s-}|^p \right] dt. \tag{26}
\end{aligned}$$

Jensen's inequality estimates the remaining summands. We get

$$J_2 \leq (2C_6)^p \int_{\mathbb{R}^r} e^{C_6 p \|y\|} \|y\|^{2p} \nu(dy) T^{p-1} \int_0^T \mathbb{E} \left[\sup_{s \in [0, t]} |u_{s-}^\varepsilon - u_{s-}|^p \right] dt$$

$$= (2C_6C_{10})^p T^{p-1} \int_0^T \mathbb{E} \left[\sup_{s \in [0, t]} |u_{s-}^\varepsilon - u_{s-}|^p \right] dt \quad (27)$$

$$\begin{aligned} J_3 &\leq (2C_1C_6C_7\varepsilon T)^p \left\{ T^{\frac{p}{2}} \left| \int_{\mathbb{R}^r} e^{2C_6\|y\|} \|y\|^4 \nu(dy) \right|^{\frac{p}{2}} + T \int_{\mathbb{R}^r} e^{pC_6\|y\|} \|y\|^p \nu(dy) \right\} \\ &\leq (2C_1C_6C_7)^p (C_{11} + C_{10}) \varepsilon^p \left\{ T^{\frac{3p}{2}} + T^{p+1} \right\} \end{aligned} \quad (28)$$

$$J_4 \leq (2C_1C_6C_7)^p C_{12} \varepsilon^p T^{2p} \quad (29)$$

Taking the supremum and expectation in inequality (19) we combine (21), (23) and (24) with (26), (27), (28) and (29). Further we use $p+1 \leq \frac{3p}{2} \leq 2p$ and $0 \leq \frac{p}{2} - 1 \leq p-1$ and obtain positive constants C_{13} and C_{14}

$$\begin{aligned} &\mathbb{E} \left[\sup_{t \in [0, T]} |u_t^\varepsilon - u_t|^p \right] \\ &\leq C_{13}(T^{2p} + T^{p+1}) \varepsilon^p + C_{14} (T^{p-1} + 1) \int_0^T \mathbb{E} \left[\sup_{s \in [0, t]} |u_s^\varepsilon - u_s|^p \right] dt \\ &=: a_\varepsilon(T) + b(T) \int_0^T \mathbb{E} \left[\sup_{s \in [0, t]} |u_s^\varepsilon - u_s|^p \right] dt. \end{aligned}$$

A standard integral version of Gronwall's inequality, as stated for instance in Lemma D.2 in [15], yields that

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |u_t^\varepsilon - u_t|^p \right] &\leq a_\varepsilon(T) (1 + b(T)T \exp(b(T)T)) \\ &\leq C_{13} T^{p+1} (1 + T^{p-1}) \varepsilon^p \left[1 + C_{14} T (1 + T^{p-1}) \exp(C_{14} T (1 + T^{p-1})) \right] \\ &\leq C_{13} T^p (1 + T)^p \varepsilon^p \exp(C_{15} T^p). \end{aligned}$$

Hence

$$\left(\mathbb{E} \left[\sup_{t \in [0, T]} |u_t^\varepsilon - u_t|^p \right] \right)^{\frac{1}{p}} \leq C_{13} \varepsilon T (1 + T) \exp(C_{15} T^p). \quad (30)$$

Eventually Minkowski's inequality and the estimates (15), (18) and (30) yield the desired result

$$\begin{aligned} &\left(\mathbb{E} \left[\sup_{s \leq T \wedge \tau^\varepsilon} |\Psi(X_t^\varepsilon(x_0)) - \Psi(X_t(x_0))|^p \right] \right)^{\frac{1}{p}} \\ &\leq C_0 \left(\mathbb{E} \left[\sup_{s \leq T \wedge \tau^\varepsilon} |u_s^\varepsilon - u_s|^p \right] \right)^{\frac{1}{p}} + C_0 \left(\mathbb{E} \left[\sup_{s \leq T \wedge \tau^\varepsilon} |v_s^\varepsilon - v_s|^p \right] \right)^{\frac{1}{p}} \\ &\leq C_0 C_{13} \varepsilon T (1 + T) \exp(C_{15} T^p) + C_0 C_1 T \varepsilon \\ &\leq C_{16} \varepsilon T \exp(C_{17} T^p). \end{aligned}$$

This finishes the proof.

■
If Z has a continuous component, the solution of equation (1) also contains a continuous Stratonovich component, see [10]. Combining the proof above with the proof of Lemma 2.1 in [5] we conclude the following.

Corollary 3.2 *Let Z be a Lévy process with characteristic triplet (b, ν, A) in \mathbb{R}^r for a drift vector $b \in \mathbb{R}^r$ and the covariance matrix A and ν as in Proposition 3.1. Then estimate (14) of Proposition 3.1 holds true with an appropriate choice of the constants k_1 and k_2 .*

Proof: The estimates of $\mathbb{E} \left[\sup_{t \in [0, T]} |u_t^\varepsilon - u_t|^p \right]$ in both cases -continuous and pure jumps- just before applying Gronwall's lemma yields polynomial estimates in T and ε of the same degree. Hence Gronwall's inequality guarantees the same estimates modulo constants. ■

4 Averaging functions on the leaves

We recall that for a fixed $x_0 \in M$ and $\varepsilon > 0$, τ^ε denotes the exit time of $X^\varepsilon(x_0)$ from the open neighbourhood $U \subset M$, which is diffeomorphic to $L_{x_0} \times V$.

Proposition 4.1 *Given $\Psi : M \rightarrow \mathbb{R}$ differentiable and $Q^\Psi : V \rightarrow \mathbb{R}$ its average on the leaves given by formula (6). For $t \geq 0$ we denote by*

$$\delta^\Psi(\varepsilon, t) := \int_0^{t \wedge \varepsilon \tau^\varepsilon} \Psi(X_{\frac{r}{\varepsilon}}^\varepsilon(x_0)) - Q^\Psi(\Pi(X_{\frac{r}{\varepsilon}}^\varepsilon(x_0))) dr.$$

Then $\delta^\Psi(\varepsilon, t)$ tends to zero, when t or ε tend to zero. Moreover, if Q^Ψ is α -Hölder continuous with $\alpha > 0$, then for any $\lambda < \alpha$, $p \geq 2$ and any $\beta \in (0, \frac{1}{2})$ we have the following estimate

$$\left(\mathbb{E} \left[\sup_{s \leq t} |\delta^\Psi(\varepsilon, s)|^p \right] \right)^{\frac{1}{p}} \leq t \left[\varepsilon^\lambda h(t, \varepsilon) + \eta \left(t |\ln \varepsilon|^{\frac{2\beta}{p}} \right) \right],$$

where $h(t, \varepsilon)$ is continuous in (t, ε) and tends to zero when t or ε do so.

Proof: (First part.) For ε sufficiently small and $t \geq 0$ we define the partition

$$t_0 = 0 < t_1^\varepsilon < \dots < t_{N^\varepsilon}^\varepsilon \leq \frac{t}{\varepsilon} \wedge \tau^\varepsilon$$

as long as X^ε has not left U with the following step size

$$\Delta_\varepsilon := t |\ln \varepsilon|^{2\frac{\beta}{p}}$$

by

$$t_n^\varepsilon := n \Delta_\varepsilon \quad \text{for} \quad 0 \leq n \leq N^\varepsilon \quad \text{where} \quad N^\varepsilon = \lfloor (\varepsilon |\ln \varepsilon|^{2\frac{\beta}{p}})^{-1} \rfloor.$$

We now represent the first summand of δ^Ψ by

$$\int_0^{t \wedge \varepsilon \tau^\varepsilon} \Psi(X_{\frac{r}{\varepsilon}}^\varepsilon(x_0)) dr = \varepsilon \int_0^{\frac{t}{\varepsilon} \wedge \tau^\varepsilon} \Psi(X_r^\varepsilon(x_0)) dr$$

$$= \varepsilon \sum_{n=0}^{N^\varepsilon-1} \int_{t_n}^{t_{n+1}} \Psi(X_r^\varepsilon(x_0)) + \varepsilon \int_{t_n}^{\frac{t}{\varepsilon} \wedge \tau^\varepsilon} \Psi(X_r^\varepsilon(x_0)) dr.$$

We lighten notation and omit for convenience in the sequel all super and subscript ε and Ψ as well as the initial value x_0 . We denote by θ the canonical shift operator on the canonical probability space $\Omega = D(\mathbb{R}, M)$ of càdlàg functions. Let $F_t(\cdot, \omega)$ the stochastic flow of the original unperturbed system in M . The triangle inequality yields

$$|\delta^\Psi(\varepsilon, t)| \leq |A_1(t, \varepsilon)| + |A_2(t, \varepsilon)| + |A_3(t, \varepsilon)| + |A_4(t, \varepsilon)|, \quad (31)$$

where

$$\begin{aligned} A_1(t, \varepsilon) &:= \varepsilon \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} [\Psi(X_r^\varepsilon) - \Psi(F_{r-t_n}(X_{t_n}^\varepsilon, \theta_{t_n}(\omega)))] dr \\ A_2(t, \varepsilon) &:= \varepsilon \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} [\Psi(F_{r-t_n}(X_{t_n}^\varepsilon, \theta_{t_n}(\omega))) - \Delta Q(\Pi(X_{t_n}^\varepsilon))] dr \\ A_3(t, \varepsilon) &:= \sum_{n=0}^{N-1} \varepsilon \Delta Q(\Pi(X_{t_n}^\varepsilon)) - \int_0^{t \wedge \varepsilon \tau^\varepsilon} Q(\Pi(X_{\frac{r}{\varepsilon}}^\varepsilon)) dr \\ A_4(t, \varepsilon) &:= \varepsilon \int_{t_n}^{\frac{t}{\varepsilon} \wedge \tau^\varepsilon} \Psi(X_r^\varepsilon(x_0)) dr. \end{aligned}$$

The following four lemmas estimate the preceding terms. This being done the proof is finished.

Lemma 4.2 *For any $\gamma \in (0, 1)$ there exists a function $h_1 = h_1(\gamma)$ such that*

$$\left(\mathbb{E} \left[\sup_{s \leq t} |A_1(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq k_1 t \varepsilon^\gamma h_1(t, \varepsilon),$$

where h_1 is continuous in ε and t and tends to zero when ε and t do so.

Proof: The proof is identical to Lemma 3.2 in [5], since Proposition 3.1 provides the same asymptotic bounds as Lemma 2.1 in [5], which enters here. Furthermore, only the Markov property of the solutions of equation (5) is exploited. \blacksquare

Lemma 4.3 *Let $\eta(t)$ be the rate of time convergence as defined in expression (9). For the process A_2 in inequality (31) we have:*

$$\left(\mathbb{E} \left[\sup_{s \leq t} |A_2(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq t \eta \left(t |\ln \varepsilon|^{-\frac{2\beta}{p}} \right).$$

Proof:

We have

$$\left(\mathbb{E} \left[\sup_{s \leq t} |A_2(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq \varepsilon \sum_{n=0}^{N-1} \left[\mathbb{E} \left| \int_{t_n}^{t_{n+1}} \Psi(F_{r-t_n}(X_{t_n}^\varepsilon, \theta_{t_n}(\omega))) dr - \Delta Q(\Pi(X_{t_n}^\varepsilon)) \right|^p \right]^{\frac{1}{p}}$$

$$= \varepsilon \Delta \sum_{n=0}^{N-1} \left[\mathbb{E} \left| \frac{1}{\Delta} \int_{t_n}^{t_{n+1}} \Psi(F_{r-t_n}(X_{t_n}^\varepsilon, \theta_{t_n}(\omega))) dr - Q(\Pi(X_{t_n}^\varepsilon)) \right|^p \right]^{\frac{1}{p}}.$$

For all $n = 0, \dots, N-1$, by the ergodic theorem, the two terms inside the modulus converges to each other when Δ goes to infinity with rate of convergence bounded by $\eta(\Delta)$. Hence, for small ε we have

$$\begin{aligned} \left(\mathbb{E} \left[\sup_{s \leq t} |A_2(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} &\leq \varepsilon N \Delta \eta(\Delta) \\ &\leq \varepsilon \left[\varepsilon^{-1} |\ln \varepsilon|^{-\frac{2\beta}{p}} \right] t |\ln \varepsilon|^{\frac{2\beta}{p}} \eta \left(t |\ln \varepsilon|^{\frac{2\beta}{p}} \right) \\ &= t \eta \left(t |\ln \varepsilon|^{\frac{2\beta}{p}} \right). \end{aligned}$$

■

Lemma 4.4 *Assume that Q^Ψ is α -Hölder continuous with $\alpha > 0$. Then the process A_3 in inequality (31) satisfies*

$$\left(\mathbb{E} \left[\sup_{s \leq t} |A_3(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq K_2 t^{1+\alpha} \varepsilon^\alpha |\ln(\varepsilon)|^{\frac{2\alpha\beta}{p}}$$

for a positive constant $K_2 > 0$.

Proof: We lighten notation $Q = Q^\Psi$. We consider the interval $[0, t]$ with the partition $0 < \varepsilon t_1 < \dots < \varepsilon t_N \leq t$

$$\begin{aligned} |A_3(t, \varepsilon)| &= \left| \sum_{n=0}^{N-1} \varepsilon \Delta Q(\Pi(X_{t_n}^\varepsilon)) - \int_0^{t \wedge \varepsilon \tau^\varepsilon} Q(\Pi(X_{\frac{r}{\varepsilon}}^\varepsilon)) dr \right| \\ &\leq \varepsilon \sum_{n=0}^{N-1} \Delta \sup_{\varepsilon t_n \leq s < \varepsilon t_{n+1}} |Q(\Pi(X_s^\varepsilon)) - Q(\Pi(X_{t_n}^\varepsilon))| \\ &\leq \varepsilon C_1 \Delta N \sup_{\varepsilon t_n \leq s < \varepsilon t_{n+1}} |v_s^\varepsilon - v_{t_n}^\varepsilon|^\alpha \\ &\leq \varepsilon C_2 \Delta N (\varepsilon h)^\alpha \\ &\leq K_2 \varepsilon^{1+\alpha} t^{1+\alpha} |\ln(\varepsilon)|^{(1+\alpha)\frac{2\beta}{p}} \varepsilon^{-1} |\ln(\varepsilon)|^{-\frac{2\beta}{p}} \\ &= K_2 t^{1+\alpha} \varepsilon^\alpha |\ln(\varepsilon)|^{\frac{2\alpha\beta}{p}}. \end{aligned}$$

■

Lemma 4.5 *The process A_4 satisfies*

$$\mathbb{E} \left[\sup_{s \leq t} |A_4(s, \varepsilon)|^p \right] \leq K_3 t \varepsilon |\ln \varepsilon|^{\frac{2\beta}{p}},$$

where $K_3 = \|\Psi\|_{\infty, U}$.

The proof follows straightforward, see also the proof of Lemma 3.5 in [5].

(Final step of Proposition 4.1) Collecting the results of the previous lemmas yields with the help of Minkowski's inequality the desired result

$$\begin{aligned} \left(\mathbb{E} \left[\sup_{s \leq t} |\delta^\Psi(\varepsilon, s)|^p \right] \right)^{\frac{1}{p}} &\leq t \left(k_1 \varepsilon^\gamma h_1(\varepsilon, t) + \eta \left(t |\ln \varepsilon|^{-\frac{2\beta}{p}} \right) + k_2 t^\alpha \varepsilon^\alpha |\ln(\varepsilon)|^{\frac{2\alpha\beta}{p}} + K_3 \varepsilon |\ln \varepsilon|^{\frac{2\beta}{p}} \right) \\ &=: t \left[\varepsilon^\lambda h(t, \varepsilon) + \eta \left(t |\ln \varepsilon|^{\frac{2\beta}{p}} \right) \right], \end{aligned}$$

where $h(t, \varepsilon)$ tends to zero if ε or t does so, for all $\lambda < \alpha$. ■

5 Proof of the main result

The gradient of each Π_i is orthogonal to the leaves. Hence by Itô's formula for canonical Marcus integrals, see e.g. [10] Proposition 4.2, we obtain for $i = 1, \dots, d$ that

$$\Pi_i(X_{\frac{t}{\varepsilon} \wedge \tau^\varepsilon}^\varepsilon) = \int_0^{\frac{t \wedge \tau^\varepsilon}{\varepsilon}} d\Pi_i(\varepsilon K)(X_r^\varepsilon) dr = \int_0^{t \wedge \tau^\varepsilon} d\Pi_i(K)(X_{\frac{r}{\varepsilon}}^\varepsilon) dr. \quad (32)$$

We may continue and change the variable

$$\begin{aligned} |\Pi_i(X_{\frac{t}{\varepsilon} \wedge \tau^\varepsilon}^\varepsilon) - w_i(t)| &\leq \int_0^{t \wedge \tau^\varepsilon} |Q^{d\Pi_i(K)}(X_{\frac{r}{\varepsilon}}^\varepsilon) - Q^{d\Pi_i(K)}(w(r))| dr + |\delta^{d\Pi_i}(t, \varepsilon)| \\ &\leq C_1 \int_0^{t \wedge \tau^\varepsilon} |\Pi_i(X_{\frac{r}{\varepsilon}}^\varepsilon) - w_i(r)| dr + |\delta^{d\Pi_i}(t, \varepsilon)| \\ &\leq C_2 \int_0^{t \wedge \tau^\varepsilon} |\Pi(X_{\frac{r}{\varepsilon}}^\varepsilon) - w(r)| dr + \sum_{i=1}^N |\delta^{d\Pi_i}(t, \varepsilon)|. \end{aligned}$$

Since the right-hand side does not depend on i , we can sum over i at the left-hand side and apply Gronwall's lemma. This yields

$$|\Pi(X_{\frac{t}{\varepsilon} \wedge \tau^\varepsilon}^\varepsilon) - w(t)| \leq e^{C_2 t} \sum_{i=1}^d |\delta^{d\Pi_i}(t, \varepsilon)|.$$

An application of Proposition 4.1 finishes the proof of the first statement. For the second part we calculate with the help of Chebyshev's inequality

$$\begin{aligned} \mathbb{P}(\varepsilon \tau^\varepsilon < T_\gamma) &\leq \mathbb{P}\left(\sup_{s \leq T_\gamma \wedge \varepsilon \tau^\varepsilon} |\Pi(X_{\frac{s}{\varepsilon} \wedge \tau^\varepsilon}^\varepsilon) - w(s)| > \gamma \right) \\ &\leq \gamma^{-p} \mathbb{E} \left[\sup_{s \leq T_\gamma \wedge \varepsilon \tau^\varepsilon} |\Pi(X_{\frac{s}{\varepsilon} \wedge \tau^\varepsilon}^\varepsilon) - w(s)|^p \right] \\ &\leq \gamma^{-p} T_\gamma^p \left[\varepsilon^\lambda h(T_\gamma, \varepsilon) + \eta \left(T_\gamma |\ln \varepsilon|^{-\frac{2\beta}{p}} \right) e^{CT_\gamma} \right]^p. \end{aligned}$$

6 Appendix

Lemma 6.1 *For any uniformly distributed random value in $[0, 2\pi)$ and $p \in \mathbb{C}$ the Gamma process defined in (11) satisfies*

$$\mathbb{E}[e^{ipe^{iZ_t}}] \rightarrow \mathbb{E}[e^{ipe^{iU}}] \quad \text{as } t \rightarrow \infty.$$

Proof: The marginal densities of the Gamma process are well-known to be $p_t(x) = \frac{\theta^t}{\Gamma(t)} x^{t-1} e^{-\theta x} dx$ such that

$$\mathbb{E}[e^{i(ue^{iL_t})}] = \int_{\mathbb{R}} e^{iue^x} \frac{\theta^t}{\Gamma(t)} x^{t-1} e^{-\theta x} dx.$$

Using the oddness of the imaginary part, the evenness of the real part and the 2π -periodicity of $x \mapsto e^{iue^x}$ for any $u \in \mathbb{C}$ we obtain

$$\begin{aligned} \mathbb{E}[e^{i(ue^{iL_t})}] &= 2 \sum_{n=1}^{\infty} \int_0^{2\pi} \Re e^{iue^y} \frac{\theta^t}{\Gamma(t)} (2\pi n + y)^{t-1} e^{-\theta(2\pi n + y)} dy \\ &= 2 \int_0^{2\pi} \Re e^{iue^y} \frac{\theta^t}{\Gamma(t)} \left(\sum_{n=1}^{\infty} (2\pi n + y)^{t-1} e^{-\theta(2\pi n + y)} \right) dy. \end{aligned}$$

We continue for fixed y with the density. It behaves as

$$\begin{aligned} \sum_{n=1}^{\infty} (2\pi n + y)^{t-1} e^{-\theta(2\pi n + y)} &\sim \int_0^{\infty} (2\pi v + y)^{t-1} e^{-\theta(2\pi v + y)} dv \\ &= \frac{1}{2\pi} \int_y^{\infty} z^{t-1} e^{-\theta z} dz \\ &= \frac{\Gamma(t)}{2\pi\theta^t} - \int_0^y \frac{z^{t-1} e^{-\theta z}}{2\pi} dz. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}[e^{i(ue^{iL_t})}] &= 2 \int_0^{2\pi} \Re e^{iue^y} \frac{\theta^t}{\Gamma(t)} \left(\frac{\Gamma(t)}{2\pi\theta^t} - \frac{y^t e^{-\theta y}}{2\pi} \right) dy \\ &= 2 \int_0^{2\pi} \frac{\Re e^{iue^y}}{2\pi} dy - 2 \frac{\theta^t}{\Gamma(t)} \int_0^{2\pi} e^{iue^y} \int_0^y \frac{\sigma^t e^{-\theta \sigma}}{2\pi} d\sigma dy. \end{aligned}$$

For the remainder term we use large Stirling's formula of $\Gamma(t) \approx \sqrt{2\pi t} \left(\frac{t}{e}\right)^t$ for large t and the Beppo-Levi theorem

$$\frac{1}{\Gamma(t)} \int_0^{2\pi} \Re e^{iue^y} \theta^t \int_0^y \frac{\sigma^t e^{-\theta \sigma}}{2\pi} d\sigma dy \leq \int_0^{2\pi} \Re e^{iue^y} \frac{(2\pi\theta)^t}{\Gamma(t)} dy \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

This concludes the proof. ■

Calculations: In the sequel we verify (9) for the radial component using the elementary identity $\cos^2(z) = \Re \frac{1}{2}(e^{i2z} + 1)$, $z \in \mathbb{R}$.

$$\mathbb{E} \frac{1}{t} \int_0^t d\Pi_r K(X_s(x_0)) ds = \mathbb{E} \frac{1}{t} \int_0^t r \cos^2(Z_s) ds$$

$$\begin{aligned}
&= \frac{r}{t} \int_0^t \mathbb{E} \cos^2(Z_s) ds \\
&= \frac{r}{t} \int_0^t \frac{1}{2} \Re \mathbb{E}[\exp(i2Z_s)] ds + \frac{r}{t} \int_0^t \frac{1}{2} ds \\
&= \frac{r}{t} \int_0^t \frac{1}{2} \exp(-Cs) ds + \frac{r}{2} \xrightarrow{t \nearrow \infty} \frac{r}{2}.
\end{aligned}$$

This yields the convergence in $p = 1$. For $p = 2$ we continue

$$\begin{aligned}
&\mathbb{E} \left[\left| \frac{1}{t} \int_0^t r \cos^2(Z_s) ds - \frac{r}{2} \right|^2 \right] \\
&= \mathbb{E} \left[\left(\frac{r}{t} \right)^2 \left(\int_0^t \cos^2(Z_s) ds \right)^2 - \frac{r^2}{t} \int_0^t \cos^2(Z_s) ds + \frac{r^2}{4} \right] \\
&= \left(\frac{r}{t} \right)^2 \mathbb{E} \left[\int_0^t \int_0^t \cos^2(Z_s) \cos^2(Z_\sigma) ds d\sigma \right] - \frac{r^2}{t} \int_0^t \mathbb{E}[\cos^2(Z_s)] ds + \frac{r^2}{4} \\
&= \left(\frac{r}{t} \right)^2 2 \int_0^t \int_0^\sigma \mathbb{E}[\cos^2(Z_s) \cos^2(Z_\sigma)] ds d\sigma + \frac{r^2}{t} \int_0^t \frac{1}{2} \exp(-Cs) ds - \frac{r^2}{4}.
\end{aligned}$$

Using the elementary identity $\cos^2(x) \cos^2(y) = \cos(2(x+y)) + \cos(2(x-y)) + 2 \cos(2x) + 2 \cos(2y) + 2$ we continue with the first term

$$\begin{aligned}
&\left(\frac{r}{t} \right)^2 2 \int_0^t \int_0^\sigma \mathbb{E}[\cos^2(Z_s) \cos^2(Z_\sigma)] ds d\sigma \\
&= \left(\frac{r}{t} \right)^2 \frac{1}{4} \int_0^t \int_0^\sigma \left(\mathbb{E}[e^{i2(Z_\sigma - Z_s)}] + \mathbb{E}[e^{i2(Z_\sigma + Z_s)}] + 2\mathbb{E}[e^{i2Z_\sigma}] + 2\mathbb{E}[e^{i2Z_s}] + 2 \right) ds d\sigma \\
&= \left(\frac{r}{t} \right)^2 \frac{1}{4} \int_0^t \int_0^\sigma \left(\mathbb{E}[e^{i2(Z_\sigma - Z_s)}] + \mathbb{E}[e^{i2(Z_\sigma - Z_s)}] \mathbb{E}[e^{i4Z_s}] + 2\mathbb{E}[e^{i2Z_\sigma}] + 2\mathbb{E}[e^{i2Z_s}] + 2 \right) ds d\sigma \\
&= \left(\frac{r}{t} \right)^2 \frac{1}{4} \int_0^t \int_0^\sigma \left(e^{-C(\sigma-s)} + e^{-C\sigma} e^{(C_4-C)s} + 2e^{-C\sigma} + 2e^{-Cs} + 2 \right) ds d\sigma \\
&= \left(\frac{r}{t} \right)^2 \frac{1}{4} \int_0^t \left(e^{-C\sigma} \frac{1}{C} (e^{C\sigma} - 1) + e^{-C\sigma} \frac{1}{C} (e^{(C-C_4)\sigma} - 1) + 2\sigma e^{-C\sigma} + \frac{2}{C} (1 - e^{-C\sigma}) + 2\sigma \right) d\sigma \\
&= \left(\frac{r}{t} \right)^2 \frac{1}{4} \int_0^t \left(\frac{1}{C} (1 - e^{-C\sigma}) + \frac{1}{C} (e^{-C_4\sigma} - e^{-C\sigma}) + 2\sigma e^{-C\sigma} + \frac{2}{C} (1 - e^{-C\sigma}) + 2\sigma \right) d\sigma \\
&\leq \left(\frac{r}{t} \right)^2 \frac{1}{4} (a + bt + t^2) \rightarrow \frac{r^2}{4}.
\end{aligned}$$

Combining the previous two results and taking the square root, the rate of convergence is of order $\eta(t) = C/\sqrt{t}$ as $t \nearrow \infty$.

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